Random Graph Models

1. Erdos - Reyni $G(n,p)$
2. Erdos - Reyni $G(n,m)$

ERDOS-REYNI $G(n,p)$

- n : no. of nodes
- Possible edges : ${}^nC_2$
- p : probability of an edge forming b/w 2 nodes
- $G(n,p)$
- Expected degree = $p(n-1) = c$

$$p = \frac{c}{n-1}$$

- Expected no. of edges = ${}^nC_2 \, p$

Q: $G(n,p)$ $n=10$, $p=0.2$, expected edges = ?

$${}^{10}C_2 \times 0.2 = 9$$

Q: Probability that a node has degree 5?

$$P(d_v = 5) = {}^{9}C_5 (0.2)^5 (0.8)^4 = 0.0166$$

Q: $G(n,p)$, $p = 0.2$, $n = 5$. $P(\text{edges} = 5) = ?$

total possible edges $= {}^{n}C_2 = {}^{5}C_2 = 10$

$P(\text{edges} = 5)$

$${}^{10}C_5 (0.2)^5 (0.8)^5 = 0.0264$$

Q: $P(\text{edges} = 5) = ?$ for $n = 10$

$${}^{10}C_2 = 45$$

$${}^{45}C_5 (0.2)^5 (0.8)^{40} = 0.052$$

Poisson Distribution

As $n \to \infty$, probability of degree k

$$f(k;\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Exponential (not polynomial) decay

ERDOS-REYNI $G(n,m)$

- n: no. of nodes
- m: no. of edges
- Let Ω = set of graphs with n nodes, m edges

total possible edges

$$|\Omega| = \binom{\binom{n}{2}}{m}$$

of which m

- Uniformly select a graph with probability $\frac{1}{|\Omega|}$

Phase Transition

- Point where diameter value starts to shrink in random graph
- Happens when average node degree $c = 1$

Q: $G(n,p)$ $p=0.2$, $n=5$. What is p when phase transition starts?

$$c = p(n-1) = p(4) = 1$$

$$p = \frac{1}{4} = 0.25$$

Q: $G(n,1)$, largest connected component size?

$$p = 1 \Rightarrow \text{size} = n$$

Local Clustering Coefficient

- Expected cc for $G(n,p) = p$

Global Clustering Coefficient

- Expected to be p

Q: $G(n,1)$ for $G(n,p)$; expected $CC = ?$

$$CC = 1$$

Shortcomings & Advantages

- Incapable of high CC networks
- Good for avg path length

Contrasting Scale-Free Nets

- Tail of DD (high k)

① Small k
- power law > ER / Poisson

② k near average k ($\langle k \rangle$)
- Poission > power

③ Large k
- power > Poisson

Disruption

- Random % of nodes removed

1) Random: diameter monotonically ↑
2) Scale-free: diameter almost same

Attack

- Willfully remove % of nodes

1) Random: diameter monotonically ↑
2) Scale-free: diameter 2x for every 5% removal (disease containment)

Barabasi-Albert Preferential-Attachment Model

- New node: prob of connecting to an existing node $\propto$ node's degree
- Rich get richer
- Initially m_0 nodes and c edges ($c < m_0$) (min degree 1)
- Average degree $= c$
- Add one node at a time, each node gets to connect to m other nodes ($m \le m_0$)
- $P \propto$ degree of i
- Probability of connecting to v_i

$$P(v_i) = \frac{d_i}{\sum_j d_j}$$

Q: BA model. New node x enters at t and makes an edge with one of the existing nodes. P(makes edge with C) = ?

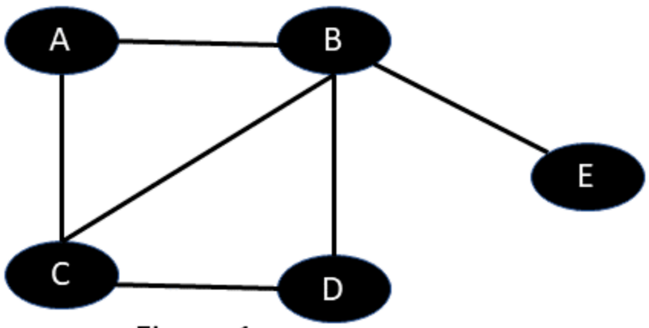


Figure 1

$P(C) = \frac{3}{6 \times 2} = \frac{1}{4}$ $\left(P(C) \propto 3 \text{ and total degree} = 2 \times \text{edges}\right)$

Q: BA model. New node x enters at t and makes an edge with one of the existing nodes. Most and least probable nodes?

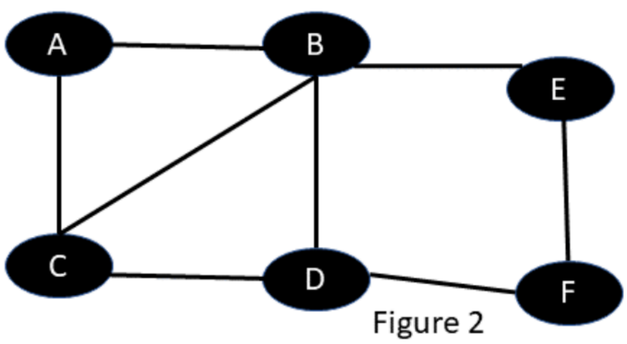


Figure 2

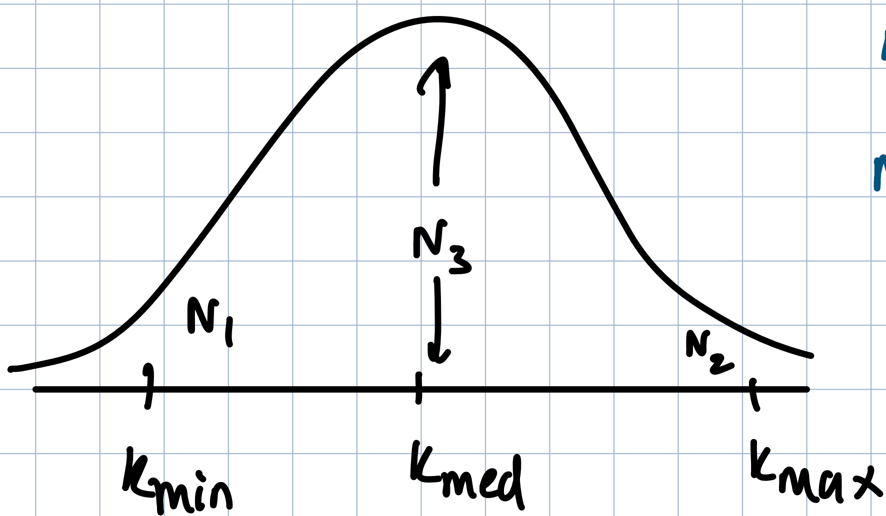
Most: B

Least: A, E, F

Shortcomings & Advantages

- Incapable of high CC
- Realistic DD
- Small avg path lengths

Q: $G(n,p) = G(1000, 0.1)$



N_1 = no. of nodes with least degree
N_2 = no. of nodes with highest degree
N_3 = no. of nodes with median degree

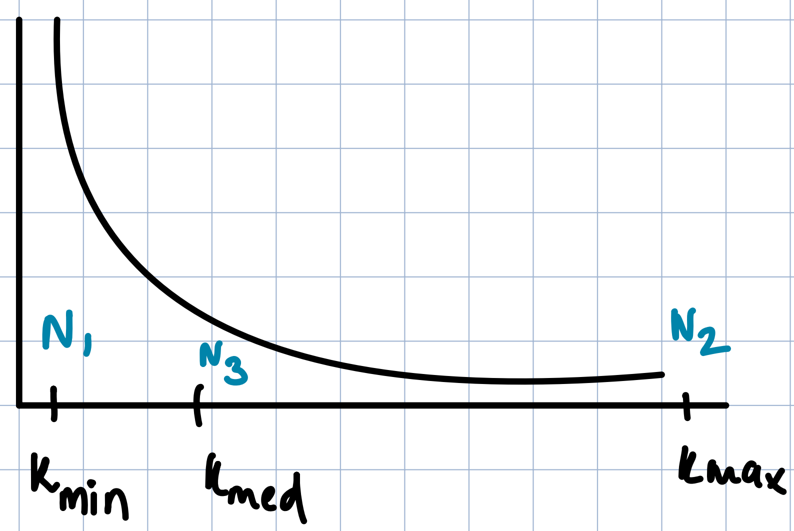
least degree = ?
largest degree = ?

Binomial dist

$N_1 < N_3$ and $N_2 < N_3$

Least = 0
Largest = 999

Q: If WWW instead, correct $N_1 < N_3$ and $N_2 < N_3$



$$N_1 > N_3 > N_2$$

Watts and Strogatz Small World Model

- Short avg path length
- High CC

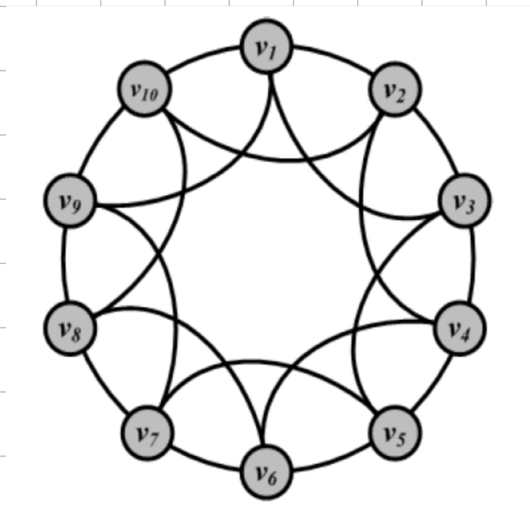
- Unrealistic but useful assumption: fixed degree (regular network)

- Regular lattice: special ring-like regular nets

- Add/move random edges to create low density of shortcuts

Regular Lattice of Degree c

- Nodes connected to prev c/2 and following c/2 neighbours

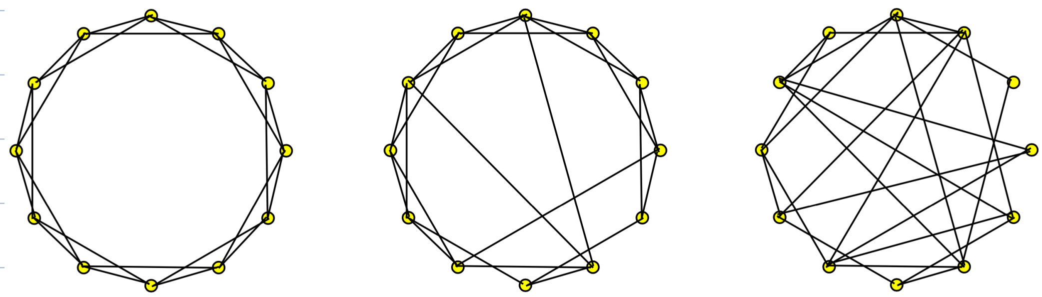


Parameters

- N: no of nodes
- K: degree of each node initially
- p: rewiring probability (no duplicates/self)

Rewiring

- Go to each edge in turn
- Move one end of that edge to a new uniformly chosen location (prob p)
- nkp shortcuts



$\beta = 0$	$\beta = 0.125$	$\beta = 1$
People know their neighbors.	People know their neighbors, and a few distant people.	People know others at random.
Clustered, but not a “small world”	Clustered and “small world”	Not clustered, but “small world”

Shortcomings & Advantages

- High CC
- Incapable of realistic DD
- Small avg path lengths

Model	Degree distribution (power law)	High average clustering coefficient	Small average path length
Random Graph model	No	No	Yes
Small world model	No	Yes	Yes
Preferential Attachment Model	Yes	No	Yes